

# An exact static two-mass solution using Nariai spacetime

Michael Fennen<sup>1</sup> and Domenico Giulini<sup>1,2</sup>

<sup>1</sup>Center for Applied Space Technology and Microgravity (ZARM)  
University of Bremen, Germany

<sup>2</sup>Institute for Theoretical Physics  
Riemann Center for Geometry and Physics  
Leibniz University of Hannover, Germany

E-mail: michael.fennen@zarm.uni-bremen.de  
giulini@itp.uni-hannover.de

**Abstract.** We show the existence of static, spherically symmetric spacetimes containing two stars of incompressible matter, possibly oppositely charged. The stars are held apart by the negative pressure of a positive cosmological constant but there is no cosmological horizon separating them. The spacetime between the stars is given by the Nariai solution, or a slight generalisation thereof in the charged case.

## 1. Introduction

Solutions to the Einstein equations with conventional matter (obeying reasonable energy conditions) representing two stars cannot be expected to be both, static and regular. After all, the stars will attract each other, so that without any agent keeping them apart they must inevitably start to approach. Staticity can only be enforced if one allows for singularities in the geometry outside the stars, usually either along the line connecting the stars and/or lines connecting each star to infinity (along the axis, in an axisymmetric situation). These singular lines can be interpreted as “struts” or “cords” that by their compressive or tensile stresses hold the stars in place. Many solutions showing this structure are known; see [1] and §35 of [27] for early discussions and chapter 10 of [9] for a comprehensive modern text-book account.

Alternatively, one may ask whether the stars could be held apart by the large negative pressure of a positive cosmological constant. Clearly, such a scenario is not meant to apply to realistic pairs of stars in our universe, but as a matter of principle concerning the study of exact solutions to Einstein’s equation it seems an obvious question to ask.

In fact, already in 1922 the mathematician and engineer Erich Trefftz attempted to find a static solution to Einstein’s vacuum equations with cosmological constant representing two “point masses” [24]. That solution was, in fact, locally identical to the Schwarzschild - De Sitter solution, also known as Kottler solution (due to [14]), that we now interpret as a single, spherically symmetric uncharged black hole in the De Sitter universe. Trefftz’ interpretation was different. He looked upon this solution as describing two “point masses” (i.e. black holes) placed at antipodal points of a 3-sphere. In this way he could maintain full spherical symmetry for the two-body situation, and not just axisymmetry, as would be the case if the two masses were not placed at antipodal points of the Universe. Einstein immediately reacted to Trefftz’ paper [13] (Doc. 387, pp. 595-596) by pointing out that the surface area of the spheres of symmetry (i.e. the  $SO(3)$  orbits) must assume a stationary value somewhere in the region exterior to the stars and that this implies, according to the field equations, that the time-time component of the metric assumes the value zero. This, according to Einstein’s interpretation, was the signal of a true and intolerable singularity.‡

Today we know better: Einstein’s “singularity” is a mere Killing horizon that shows the existence of a cosmological event horizon (in agreement with Hawking’s strong rigidity theorem) in the region between the two bodies. In the present paper we wish to address the question of whether we can use the cosmological constant to support to massive stars at finite distance in static equilibrium *without* any cosmological horizon separating them.

‡ Quite surprisingly, initially even Hermann Weyl in [26] followed Einstein’s belief that a non-vanishing cosmological constant would prevent the existence of vacuum solutions (“es widerstreitet dem Einstein’schen Gesetz, dass die Welt vollständig masseleer ist”; [26]) and that in static spacetimes with Killing vector  $\partial/\partial t$  the vanishing of  $\mathbf{g}(\partial/\partial t, \partial/\partial t)$  signals a singularity.

## 2. Spherical symmetry

In order to evade the conclusions of Einstein's argument we need to identify its mathematical origin. This is easy to do if one takes due care of the restrictions imposed by spherical symmetry. Recall that a spacetime  $(M, g)$  is spherically symmetric if there exists an action of  $SO(3)$  by isometries whose orbits are spacelike 2-spheres. These spheres can then be parametrised in the usual fashion by spherical polar coordinates  $\vartheta$  and  $\varphi$ . Moreover, the normal bundle to these orbits can be shown to be integrable (a step usually omitted in most textbooks; an exception is [23], section 4.10.1). This means that the distribution of (Lorentzian) 2-planes perpendicular to the distribution of (Euclidean) 2-planes tangent to the  $SO(3)$  orbits are locally integrable. Consequently, the metric can be locally parametrised by coordinates  $(t, r, \vartheta, \varphi)$  with no off-diagonal metric coefficients between  $(t, r)$  and  $(\vartheta, \varphi)$ .

Note that spherical symmetry alone implies other useful structural elements. For example, there is a preferred function,  $R$ , whose value at each point  $p \in M$  equals  $\sqrt{A(p)/4\pi}$ , where  $A(p)$  is the area of the  $SO(3)$  orbit through  $p$ . This function  $R$  is called the *areal radius*. We can use  $R$  as radius coordinate in regions where the one-form  $\mathbf{d}R$  nowhere vanishes. Suppose this being the case and that  $\mathbf{d}R$  is spacelike, i.e. the 3-dimensional sub-manifolds of constant  $R$  are timelike. Then, up to normalisation, there is a unique  $SO(3)$ -invariant timelike vector field (necessarily orthogonal to the  $SO(3)$  orbits) that is annihilated by  $\mathbf{d}R$ . It is not difficult to see that this vector field is hypersurface orthogonal and hence gives rise to a preferred foliation of spacetime by spacelike hypersurfaces.<sup>§</sup> Choosing a time function  $t$  whose level sets are these hypersurfaces, the metric takes the form

$$\mathbf{g} = -a(t, R) \mathbf{d}t^2 + r(t, R) \mathbf{d}R^2 + R^2(\mathbf{d}\vartheta^2 + \sin^2 \vartheta \mathbf{d}\varphi^2). \quad (1)$$

Similarly, the other two cases for non-vanishing  $\mathbf{d}R \neq 0$ , where  $\mathbf{d}R$  is time- or lightlike, can also be written down.

Often (1) (together with the other two cases) are taken to exhaust the “general forms” of spherically symmetric metrics. But there is still the case left where  $\mathbf{d}R$  is neither spacelike, timelike, or lightlike, but simply vanishes, at least locally. In this case we cannot use  $R$  as coordinate. Let us now focus on these somewhat exceptional cases and see under what conditions they occur as solutions to Einstein's equations, restricting attention to the static case. Then the metric can then be written in the form

$$\mathbf{g} = -a^2(z) \mathbf{d}t^2 + \mathbf{d}z^2 + R^2(z)(\mathbf{d}\vartheta^2 + \sin^2 \vartheta \mathbf{d}\varphi^2), \quad (2)$$

where we now used a radial coordinate  $z$  (called  $z$  rather than  $r$  for reasons to become clear soon) such that  $\mathbf{g}(\partial_z, \partial_z) = 1$ . The areal radius  $R$  is now a function of  $z$  that may well have stationary points. The non-vanishing components of the left-hand side of Einstein's equations,  $G_{\mu\nu} + \Lambda g_{\mu\nu}$ , with respect to the orthonormal co-frame

<sup>§</sup> This vector field is sometimes referred to as *Kodama field*; e.g., [7].

$$\theta^0 = a \mathbf{d}t, \quad (3a)$$

$$\theta^1 = \mathbf{d}z, \quad (3b)$$

$$\theta^2 = R \mathbf{d}\vartheta, \quad (3c)$$

$$\theta^3 = R \sin \vartheta \mathbf{d}\varphi, \quad (3d)$$

are then given by (see Appendix (B.17)-(B.19) and note that  $1 = -g_{00} = g_{11} = g_{22} = g_{33}$ )

$$G_{00} - \Lambda = -2 \frac{R''}{R} + \frac{1 - R'^2}{R^2} - \Lambda, \quad (4a)$$

$$G_{11} + \Lambda = 2 \frac{a' R'}{a R} - \frac{1 - R'^2}{R^2} + \Lambda, \quad (4b)$$

$$G_{22} + \Lambda = \frac{a''}{a} + \frac{R''}{R} + \frac{a' R'}{a R} + \Lambda. \quad (4c)$$

We did not write down the 33-component, which is identical to the 22-component due to spherical symmetry.

Let us focus on solutions to Einstein's equations with cosmological constant and vanishing energy-momentum tensor (vacuum solutions). Einstein's equations are then equivalent to each of the three expressions above being equal to zero. Taking the sum of the first two expressions (4a) and (4b) and equating it to zero gives

$$a R'' = a' R'. \quad (5)$$

Now suppose  $z = z_*$  is a stationary point for  $R$ , i.e.  $R'(z_*) = 0$ . Then (5) shows that  $a(z_*) = 0$  if  $R''(z_*) \neq 0$ , i.e. if  $R$  assumes a proper extremal value at  $z_*$ . But zeros of  $a$  correspond to Killing horizons, which is just Einstein's observation (in modern terminology and interpretation). But now it is also clear how to avoid this conclusion (of a Killing horizon), namely to assume that  $R$  is constant;  $R = \mathcal{R}_N$ . Either of (4a) or (4b) then gives

$$\mathcal{R}_N = 1/\sqrt{\Lambda}, \quad (6)$$

which shows that we have to assume  $\Lambda > 0$ . The final equation (4c) gives  $a'' = -\Lambda a$ , which is the harmonic-oscillator equation. The two integration constants (amplitude and phase) can be absorbed by redefining the scale of the  $t$  and the origin of the  $r$  coordinate. This leads to the *Nariai* metric (in static form), known since 1950 [20][19]:

$$\mathbf{g} = -\cos^2(z/\mathcal{R}_N) \mathbf{d}t^2 + \mathbf{d}z^2 + \mathcal{R}_N^2 (\mathbf{d}\vartheta^2 + \sin^2 \vartheta \mathbf{d}\varphi^2). \quad (7)$$

Note that the  $SO(3)$  orbits are mutually isometric 2-spheres of radius  $\mathcal{R}_N$ . This is why we called the spatial coordinate perpendicular to these orbits  $z$  rather than  $r$ , because the  $z$ -family of 2-spheres looks like a cylinder  $\mathbb{R} \times S^2$ . If we wish to avoid Killing horizons we have to restrict the cylinder to that region parametrised by  $z \in (-\mathcal{R}_N\pi/2, \mathcal{R}_N\pi/2)$ .

Finally, if we consider the Einstein's equations with matter,

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu}, \quad (8)$$

the sum of the 00 and 11 components (4a) and (4b) tell us that at a stationary point  $z_*$  for  $R$  we have  $8\pi G(T_{00} + T_{11}) = -2R''/R$ . The weak-energy condition implies that the left-hand side is non-negative, hence  $R''(z_*) \leq 0$ . This implies that  $R$  cannot have a local minimum inside a star whose matter obeys the weak energy condition.

### 3. Charged-Nariai spacetime

In this section we will briefly show how to generalise the Nariai spacetime (7) so as to include an electric field parallel to the cylinder axis which makes the two ends of the cylinder appear equally and oppositely charged. Note that we do not just solve Maxwell's equations on the background (7), but we seek a new solution to the Einstein-Maxwell equations that, in an appropriate sense, generalises (7). That generalised metric shall be of the form (cf. (2))

$$\begin{aligned} \mathbf{g}_N &= -\theta^0 \otimes \theta^0 + \sum_{a=1}^3 \theta^a \otimes \theta^a \\ &= -a^2(z) \mathbf{d}t^2 + \mathbf{d}z^2 + \mathcal{R}_N^2 \mathbf{d}\Omega^2, \end{aligned} \quad (9)$$

with a yet unspecified function  $a(z)$ . The electric field is also assumed to be spherically symmetric and spacelike, which implies that tangent to the simultaneity hypersurfaces  $dt = 0$  it points parallel to the normal of the 2-sphere orbits of  $SO(3)$  and that its modulus depends only on  $z$ . Hence, without loss of generality, the electromagnetic 2-form is given by

$$\mathbf{F} = -E(z) a(z) \mathbf{d}t \wedge \mathbf{d}z = \frac{q}{d^2(z)} a(z) \mathbf{d}t \wedge \mathbf{d}z. \quad (10)$$

Here  $q$  is a constant and  $d$  is a real-valued function with physical dimension of length. The signs are chosen such that positive  $q$  correspond to electric fields pointing in negative  $z$  direction, as will become apparent below.

In the absence of sources Maxwell's equations read  $\mathbf{d}\mathbf{F} = 0$  and  $\mathbf{d} * \mathbf{F} = 0$ . The first equation is solved by  $\mathbf{F} = \mathbf{d}\mathbf{A}$  with  $\mathbf{A} = -\Phi(z) \mathbf{d}t$  and

$$\Phi(z) = \int_{z_0}^z \frac{q}{d^2(x)} a(x) \mathbf{d}x. \quad (11)$$

The Hodge-duality map  $*$  is defined with respect to the space-time orientation represented by the volume form  $\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$ , so that, e.g.,  $*(\theta^0 \wedge \theta^1) = -\theta^2 \wedge \theta^3$ . Hence

$$* \mathbf{F} = -\frac{q}{d^2(z)} \mathcal{R}_N^2 \sin \vartheta \mathbf{d}\vartheta \wedge \mathbf{d}\varphi = \frac{q}{d^2(z)} \mathcal{R}_N^2 \mathbf{d}(\cos \vartheta \mathbf{d}\varphi). \quad (12)$$

This shows that Maxwell's second equation is equivalent to  $d(z)$  being constant. Since the two constants  $q$  and  $d$  only appear in the combination  $q/d^2$ , we may without loss of generality set  $d(z) \equiv \mathcal{R}_N$ . The flux through any of the  $SO(3)$  orbits therefore equals

$$Q = \frac{1}{4\pi} \int_{S^2} * \mathbf{F} = \pm q, \quad (13)$$

where the sign on the right-hand side depends on the orientation of  $S^2$ . If the orientation of spacetime is represented by the volume form  $\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$ , the orientation of space (the level sets of  $t$ ) is represented by  $\theta^1 \wedge \theta^2 \wedge \theta^3$  if  $\partial/\partial t$  is taken to point *outward* in spacetime. Furthermore, the orientation of the  $SO(3)$  orbits is represented by  $\theta^2 \wedge \theta^3$  if  $\partial/\partial z$  is taken to point *outward* in space. Then  $Q = -q$  which for  $q > 0$  means that the negative end ( $z < 0$ ) is negatively and the positive end ( $z > 0$ ) is positively charged. Accordingly, the electric field points into the negative  $z$  direction, as stated above.

### 3.1. Einsteins's equations

Next we evaluate Einstein's equation, using the ansatz (9) for its left-hand side and (10) for its right-hand side. The right-hand side is given by the stress-energy tensor of an electromagnetic field, which is

$$T_{\mu\nu}^{(\text{em})} = \frac{1}{4\pi} \left( g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right), \quad (14)$$

where  $F_{\mu\nu}$  are the components of the electromagnetic field tensor (10). With respect to the frame (2) the only non-vanishing component is  $F_{01} = q/\mathcal{R}_N^2$ . Hence the electromagnetic stress-energy tensor has no off-diagonal components and reads  $(q/\mathcal{R}_N^2)^2 (8\pi)^{-1} \text{diag}(1, -1, 1, 1)$ . Writing  $8\pi G$  times these on the right-hand side of (4a) and (4c) (due to  $R' = R'' = 0$  the 00- and 11 components become identical) Einstein's equations turn out to be identical to the following two equations:

$$\frac{1}{\mathcal{R}_N^2} - \Lambda = \frac{Gq^2}{\mathcal{R}_N^4}, \quad (15a)$$

$$\frac{a''(z)}{a(z)} + \Lambda = \frac{Gq^2}{\mathcal{R}_N^4}. \quad (15b)$$

The first equation (15a) merely relates the three constants  $\Lambda$ ,  $\mathcal{R}_N$ , and  $q^2$ . In particular, it allows to express the cosmological constant  $\Lambda$  in terms of the radius  $\mathcal{R}_N$  of the orbit 2-spheres and the charge  $q$ . Using this to eliminate  $\Lambda$  in (15b) we get

$$a''(z) + \frac{1}{\mathcal{R}_N^2} \left( 1 - \frac{2Gq^2}{\mathcal{R}_N^2} \right) a(z) = 0. \quad (16)$$

Depending on the ratio of the charge to the radius, the solutions to (16) are:

$$a(z) = \begin{cases} \mathcal{C} \cos\left(\frac{\lambda z}{\mathcal{R}_N}\right) + \mathcal{D} \sin\left(\frac{\lambda z}{\mathcal{R}_N}\right) & \text{if } q^2 < \frac{\mathcal{R}_N^2}{2G}, \\ \mathcal{C} + \mathcal{D}z & \text{if } q^2 = \frac{\mathcal{R}_N^2}{2G}, \\ \mathcal{C} \cosh\left(\frac{\lambda z}{\mathcal{R}_N}\right) + \mathcal{D} \sinh\left(\frac{\lambda z}{\mathcal{R}_N}\right) & \text{if } q^2 > \frac{\mathcal{R}_N^2}{2G}, \end{cases} \quad (17)$$

where  $\lambda^2 = \left| 1 - \frac{2Gq^2}{\mathcal{R}_N^2} \right|$  and  $\mathcal{C}$  and  $\mathcal{D}$  are integration constants.

In the following we will choose  $\mathcal{D} = 0$  so as to make the solutions invariant under the reflection  $z \mapsto -z$ . Without loss of generality we may then further set  $\mathcal{C} = 1$ , since

this can always be achieved by a constant rescaling of the time coordinate. The metric

$$\mathbf{g}_N = -\cos^2\left(\frac{\lambda z}{\mathcal{R}_N}\right) \mathbf{d}t^2 + \mathbf{d}z^2 + \mathcal{R}_N^2 \mathbf{d}\Omega^2 \quad (18)$$

may be called a charged-Nariai spacetime. It reduces to the ordinary Nariai spacetime in the limit of vanishing charge and will play an important rôle in what is to follow. Killing horizons are now absent if we restrict to  $z \in (-\mathcal{R}_N\pi/2\lambda, \mathcal{R}_N\pi/2\lambda)$ .

The third case,

$$\mathbf{g}_{BR} = -\cosh^2\left(\frac{\lambda z}{\mathcal{R}_N}\right) \mathbf{d}t^2 + \mathbf{d}z^2 + \mathcal{R}_N^2 \mathbf{d}\Omega^2, \quad (19)$$

is the Bertotti-Robinson metric [2] generalised to non-zero cosmological constant, sometimes also referred to as cosmological Bertotti-Robinson spacetime [9]. The special value  $q^2 = \frac{\mathcal{R}_N^2}{G}$ , i.e.  $\lambda = 1$ , corresponds to vanishing cosmological constant.

Finally we remark on the notion of “mass”. In spherically symmetric spacetimes there is, next to the areal radius  $R$ , another geometrically defined function, which is the sectional curvature of spacetime tangent to the  $SO(3)$  orbits. (Note that this is generally *not* the Gaussian curvature of the orbit, unless the orbit is totally geodesic.) Since the orbits foliate spacetime, this defines a function on spacetime by assigning to each point the sectional curvature tangent to the orbit through it. Clearly this function is constant on each orbit. Writing the metric in the form (2) this function is just given by the  $R_{2323}$  component of the Riemann tensor with respect to the frame (3a)-(3d), which we calculated in (B.11). Note that the sectional curvature of spacetime tangent to the group orbits equals the Gaussian curvature of the 2-dimensional surface locally spanned by all spacetime geodesics starting tangentially to the orbit. This surface touches the orbit to first, but generally not to second order. Consequently, their Gaussian curvatures are generally not the same, as can be clearly seen from expression (B.11), which differs from  $R^{-2}$  (the Gaussian curvature of the orbits) by the  $-R''/R^2$  term, which is non zero iff  $R' \neq 0$ . Hence, in the case of constant  $R$ , the spacetime’s sectional curvature tangent to the orbits is identical to their Gaussian curvature.

Now, if we multiply this sectional curvature by half the third power of the areal radius we get another function on spacetime which is constant on the orbits and which equals the Misner-Sharp mass in geometric units. (The Misner-Sharp mass was first introduced in a non-geometric fashion in [17]. Its geometric definition is discussed, e.g., in [7]). It can be shown [7] to equal the Hawking mass [10] and, according to (B.11), has the following simple form (in physical units where  $c = 1$ )

$$M = \frac{R}{2G} \left(1 - g^{-1}(\mathbf{d}R, \mathbf{d}R)\right) = \frac{\mathcal{R}_N}{2G}. \quad (20)$$

where the first expression is the generally valid one if  $R$  denotes the areal radius, and the second expression is valid for  $dR = 0$ . We conclude that the charged-Nariai spacetime is almost determined by the mass and charge. “Almost” but not quite completely, because we also need to indicate the range of the coordinate  $z$ , i.e. the length of the cylinder. This makes the charged Nariai solution a three-parameter family.

A star matched to the Nariai metric must have the very same mass as a result of the matching conditions. Two stars matched to the Nariai cylinder, one at each end, must clearly have equal and opposite charge. Once the mass and charge are fixed the only degree of freedom that is left is the position of the stars, i.e., the length of the cylinder. We expect that the position should be related to the pressure within the star, a property not yet used. To see this, we first need to find the star's interior solutions.

#### 4. Charged star solution

In this section we wish to derive simple solutions for spacetime regions interior to the star. We will take the distribution of bare rest-mass,  $\rho$ , to be constant (with respect to the proper geometric measure induced in the hypersurfaces of simultaneity) and the charge distribution similarly simple, though not proportional to the distribution of bare rest-mass. So our solution will be a generalisation of the inner Schwarzschild solution [22] to the charged case including a cosmological constant. Both cases have already been considered separately before: the neutral case with a cosmological constant in [3, 4, 5], and the charged case without cosmological constant in [15]. To our knowledge, the only treatment of spherically-symmetric static stars with charge and non-vanishing cosmological constant is given in [6]. But the equations of state used in this paper differ from our condition of constant mass-density (incompressibility). Our intended application of this solutions is also different.

As we have already seen,  $R$  cannot assume a local minimum inside a star made of matter satisfying the weak energy-condition. As the centre of the star is a fixed point of the action of  $SO(3)$ , and the metric is required to be regular inside the star,  $R$  must tend to zero as we approach the centre. Hence  $R$  is a monotonic function as the radius increases from zero until the first maximum is reached.|| As for our construction we will only be interested in stars where  $R$  assumes a maximum on its boundary, we may use  $R$  as a coordinate function inside the star, which we now call  $r$ . Hence the metric inside the star may be written in the form

$$\begin{aligned} \mathbf{g}_S &= -\theta^0 \otimes \theta^0 + \sum_{a=1}^3 \theta^a \otimes \theta^a \\ &= -e^{2a(r)} \mathbf{d}t^2 + e^{2b(r)} \mathbf{d}r^2 + r^2 \mathbf{d}\Omega^2, \end{aligned} \quad (21)$$

where now

$$\theta^0 = e^{a(r)} \mathbf{d}t, \quad (22a)$$

$$\theta^1 = e^{b(r)} \mathbf{d}r, \quad (22b)$$

$$\theta^2 = r \mathbf{d}\vartheta, \quad (22c)$$

$$\theta^3 = r \sin \vartheta \mathbf{d}\varphi. \quad (22d)$$

|| In Appendix B of [11] is allegedly shown that  $R$  cannot have any extremum inside spherically-symmetric and static stars satisfying the weak energy-condition, but that is not correct. It seems that the existence of a term corresponding to our  $R''(z_*)$  ( $Y''$  in their notation, resulting from their equation (A3)) has been overlooked.



#### 4.1. Maxwell equations

The most general static and spherically-symmetric electric field is given by the electromagnetic 2-form

$$\mathbf{F} = -E(r) \theta^0 \wedge \theta^1 = -E(r) e^{a(r)+b(r)} \mathbf{d}t \wedge \mathbf{d}r. \quad (23)$$

As before, Maxwell's first equation,  $\mathbf{d}\mathbf{F} = 0$ , is solved by  $\mathbf{F} = \mathbf{d}\mathbf{A}$ , where  $\mathbf{A} = -\Phi(r) \mathbf{d}t$  with

$$\Phi(r) = - \int_0^r dx E(x) e^{a(x)+b(x)}. \quad (24)$$

Maxwell's second (inhomogeneous) equation reads  $\mathbf{d} * \mathbf{F} = 4\pi * \mathbf{J}$ , where  $\mathbf{J} = -\sigma(r) \theta^0$  is the current-density 1-form. On the left-hand side we get

$$\mathbf{d} * \mathbf{F} = -\frac{d}{dr} (E(r) r^2) \mathbf{d}r \wedge \mathbf{d}(\cos \vartheta \mathbf{d}\varphi) \quad (25)$$

and on the right-hand side

$$4\pi * \mathbf{J} = 4\pi \sigma(r) \theta^1 \wedge \theta^2 \wedge \theta^3 = -4\pi \sigma(r) e^{b(r)} r^2 \mathbf{d}r \wedge \mathbf{d}(\cos \vartheta \mathbf{d}\varphi). \quad (26)$$

Hence the inhomogeneous Maxwell equations are equivalent to

$$\frac{d}{dr} (E(r) r^2) = 4\pi r^2 \sigma(r) e^{b(r)}, \quad (27)$$

the solution of which is readily obtained if we restrict to a particular radial charge distribution given by

$$\sigma(r) e^{b(r)} = \sigma_{\pm} = \text{const.} \quad (28)$$

Note that this does not correspond to constant charge density with respect to the proper (3-dimensional) geometric volume measure  $\theta^1 \wedge \theta^2 \wedge \theta^3$ , but rather to a constant density with respect to the “areal volume” measure  $r^2 \sin \vartheta \mathbf{d}r \wedge \mathbf{d}\vartheta \wedge \mathbf{d}\varphi$ . We will later see that, assuming a constant bare rest-mass distribution with respect to the proper geometric volume, the (active) gravitational mass (which takes into account gravitational binding energies) will also be constantly distributed with respect to the “areal volume” (compare equation (54)). Hence (28) amounts to the assumption that the densities for gravitational mass and electric charge are constant inside the star. The charge inside a ball of areal radius  $r$  now becomes

$$Q(r) = 4\pi \int_0^r dx x^2 \sigma(x) e^{b(x)} = \frac{4\pi}{3} \sigma_{\pm} r^3 \quad (29)$$

and the solution to (27) is then, clearly, just

$$E(r) = \frac{Q(r)}{r^2}. \quad (30)$$

#### 4.2. Einstein's equations

We recall that the components of the electromagnetic stress-energy tensor are given by

$$T_{\mu\nu}^{(\text{em})} = \frac{1}{4\pi} \left( g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right), \quad (31)$$

In what follows, all components refer to the orthonormal basis (22a)-(22d). In the electric and spherically-symmetric case only the  $F_{01}$  component is non-zero and given by (30), so that

$$T_{\mu\nu} = \frac{Q^2(r)}{8\pi r^4} \text{diag}(1, -1, 1, 1). \quad (32)$$

For the metric (21) and the energy-momentum Tensor above the 00, 11, and 22 components of Einstein's equation contain all the information. They read, respectively, (dropping arguments of functions for brevity)

$$\frac{1}{r^2} + \left( \frac{2b'}{r} - \frac{1}{r^2} \right) e^{-2b} - \Lambda = 2\bar{\rho} + \frac{GQ^2}{r^4}, \quad (33a)$$

$$-\frac{1}{r^2} + \left( \frac{2a'}{r} + \frac{1}{r^2} \right) e^{-2b} + \Lambda = 2\bar{p} - \frac{GQ^2}{r^4}, \quad (33b)$$

$$\left( a'' + a'^2 - a'b' + \frac{a' - b'}{r} \right) e^{-2b} + \Lambda = 2\bar{p} + \frac{GQ^2}{r^4}. \quad (33c)$$

Here and in the sequel we used the shorthand

$$\bar{X} := 4\pi GX, \quad (X = p, \rho, \sigma_{\pm}). \quad (34)$$

Stress-energy conservation,  $\nabla_{\mu} T^{\mu\nu} = 0$ , is equivalent to

$$\bar{p}'(r) + (\bar{\rho} + \bar{p}(r)) a'(r) = \frac{GQ(r)Q'(r)}{r^4}. \quad (35)$$

We can rewrite the first equation (33a) as

$$\frac{d}{dr} (r e^{-2b}) = 1 - 2\bar{\rho} r^2 - \frac{\bar{\sigma}_{\pm}^2}{9G} r^4 - \Lambda r^2 \quad (36)$$

with the solution

$$e^{-2b} = 1 - \alpha r^2 - \beta r^4, \quad (37)$$

where

$$\alpha = \frac{1}{3} (2\bar{\rho} + \Lambda), \quad (38a)$$

$$\beta = \frac{\bar{\sigma}_{\pm}^2}{45G}. \quad (38b)$$

Eliminating  $p(r)$  from (33c) using (33b) we get a differential equation for  $a(r)$ . Using  $GQ^2(r) = 5\beta r^6$  and (37) we obtain

$$(1 - \alpha r^2 - \beta r^4) \left( a'' + a'^2 \right) - \frac{a'}{r} (1 + \beta r^4) = 11\beta r^2. \quad (39)$$

This somewhat complicated non-linear second order differential equation is simplified by substituting

$$a'(r) = \frac{r}{\sqrt{1 - \alpha r^2 - \beta r^4}} \left( \sqrt{11\beta} + \frac{1}{f(r)} \right) \quad (40)$$

after which it becomes an ordinary linear first-order differential equation

$$f'(r) = \frac{r}{\sqrt{1 - \alpha r^2 - \beta r^4}} (2\sqrt{11\beta}f(r) + 1). \quad (41)$$

This is easily integrated by separation and yields

$$2\sqrt{11\beta}f(r) = \mathcal{B} \exp \left( \sqrt{11} \arcsin \left( \frac{\alpha + 2\beta r^2}{\sqrt{\alpha^2 + 4\beta}} \right) \right) - 1. \quad (42)$$

Here  $\mathcal{B}$  is an integration constant which depends on the central pressure  $p_c = p(0)$ , as we will see in the next section. We can combine the two equations (40) and (41) to

$$a'(r) = \frac{\sqrt{11\beta} + \frac{1}{f(r)}}{2\sqrt{11\beta}f(r) + 1} f'(r), \quad (43)$$

which is also easily integrated to

$$e^{2a(r)} = \frac{\mathcal{A}f(r)^2}{2\sqrt{11\beta}f(r) + 1}, \quad (44)$$

with another integration constant  $\mathcal{A}$ . This constant could be absorbed by a rescaling of the time coordinate  $t$  but we have to keep it here because we have already used the rescaling freedom in the charged-Nariai spacetime.

Altogether we get for a charged star the metric

$$\mathbf{g}_S = -\frac{\mathcal{A}f^2(r)}{2\sqrt{11\beta}f(r) + 1} dt^2 + \frac{1}{1 - \alpha r^2 - \beta r^4} dr^2 + r^2 d\Omega^2. \quad (45)$$

The pressure function is determined by the second equation (33b) as

$$\bar{p}(r) = \sqrt{1 - \alpha r^2 - \beta r^4} \left( \sqrt{11\beta} + \frac{1}{f(r)} \right) + \frac{\Lambda - \alpha}{2} + 2\beta r^2. \quad (46)$$

The radial coordinate  $r$  is only valid for  $r < \mathcal{R}_S$  where  $\mathcal{R}_S$  is the first positive root of  $e^{-2b(r)}$ . This means  $1 - \alpha\mathcal{R}_S^2 - \beta\mathcal{R}_S^4 = 0$  so that there is a coordinate singularity in the metric at  $r = \mathcal{R}_S$ . However, in the following it will be necessary that the metric is regular here because we will see that this will turn out to be the radius of our stars. Therefore we rearrange

$$e^{-2b(r)} = 1 - \alpha r^2 - \beta r^4 = \beta ((r^2 + \mathcal{S}^2)(\mathcal{R}_S^2 - r^2)), \quad (47)$$

where  $2\beta\mathcal{S}^2 = \alpha + \sqrt{\alpha^2 + 4\beta}$  and  $2\beta\mathcal{R}_S^2 = -\alpha + \sqrt{\alpha^2 + 4\beta} > 0$ . Using a new radial coordinate  $\chi$  such that  $r = \mathcal{R}_S \sin \chi$  allows us to eliminate the coordinate singularity in the metric at the equator  $\chi = \frac{\pi}{2}$  or  $r = \mathcal{R}_S$ , respectively. For  $0 \leq \chi \leq \frac{\pi}{2}$  we simply have  $F(\chi) = f(\mathcal{R}_S \sin \chi)$  or

$$2\sqrt{11\beta}F(\chi) = \mathcal{B} \exp \left( \sqrt{11} \arcsin \left( \frac{\alpha + 2\beta\mathcal{R}_S^2 \sin^2 \chi}{\sqrt{\alpha^2 + 4\beta}} \right) \right) - 1. \quad (48)$$

However, it is possible to extend the solution beyond the equator up to the second pole  $\chi = \pi$ . For  $\frac{\pi}{2} \leq \chi \leq \pi$  we get

$$2\sqrt{11\beta}F(\chi) = \mathcal{B} \exp \left( \sqrt{11} \left( \pi - \arcsin \left( \frac{\alpha + 2\beta\mathcal{R}_S^2 \sin^2 \chi}{\sqrt{\alpha^2 + 4\beta}} \right) \right) \right) - 1. \quad (49)$$

The derivative of  $F$  is given by

$$F'(\chi) = \frac{\mathcal{R}_S \sin \chi}{\sqrt{\beta (\mathcal{R}_S^2 \sin^2 \chi + \mathcal{S}^2)}} \left( 2\sqrt{11\beta}F(\chi) + 1 \right) \quad (50)$$

is regular for the whole interval  $[0, \pi]$ . So this new coordinate covers the whole (distorted) 3-sphere with the metric

$$\mathbf{g}_S = -\frac{\mathcal{A}F^2(\chi)}{2\sqrt{11\beta}F(\chi) + 1} \mathbf{d}t^2 + \frac{1}{\beta (\mathcal{R}_S^2 \sin^2 \chi + \mathcal{S}^2)} \mathbf{d}\chi^2 + \mathcal{R}_S^2 \sin^2 \chi \mathbf{d}\Omega^2. \quad (51)$$

Finally, the pressure becomes

$$\bar{P}(\chi) = \sqrt{\beta (\mathcal{R}_S^2 \sin^2 \chi + \mathcal{S}^2)} \left( \sqrt{11\beta} + \frac{1}{F(\chi)} \right) \mathcal{R}_S \cos \chi + \frac{\Lambda - \alpha}{2} + 2\beta\mathcal{R}_S^2 \sin^2 \chi. \quad (52)$$

Let us end this section with a few observations and remarks:

- (i) The function  $F(\chi)$  is strictly monotonic, increasing for  $\mathcal{B} > 0$  and decreasing for  $\mathcal{B} < 0$ .
- (ii) Using the general expression for the Misner-Sharp mass (20) applied to the metric (51), from which the areal radius immediately follows to be  $R = \mathcal{R}_S \sin \chi$ , and also taking into account the definitions (38a)-(38b), we get the following expression for the Misner-Sharp mass inside a ball of latitude  $\chi$ :

$$M(\chi) = \frac{\mathcal{R}_S^3 \sin^3 \chi}{2G} (\alpha + \beta\mathcal{R}_S^2 \sin^2 \chi). \quad (53)$$

The first term in this equation comprises the contributions from the matter and the cosmological constant. If expressed directly in terms of the parameters  $\rho$  and  $\Lambda$  and also in terms of the areal radius  $R = r$  (recall that in the coordinates in which the metric is written as (21) we have  $R = r$ ) it reads

$$M_{(\rho, \Lambda)}(r) = \frac{4\pi}{3} r^3 \left( \rho + \frac{\Lambda}{8\pi G} \right). \quad (54)$$

Note that  $\frac{4\pi}{3}r^3$  is *not* the spatial volume  $V(r)$  of the ball bound by the sphere of areal radius  $r$ . The latter is bigger and the difference  $(\frac{4\pi}{3}r^3 - V(r))\rho$  just accounts for the (negative) gravitational binding energy of the matter represented by  $\rho$ . This is well known from the ordinary inner Schwarzschild solution. Note that the contribution from the cosmological constant is likewise diminished by this volume factor. In fact, the very same is also true for the electromagnetic part: Using (38b), (34), and (29) we can easily see that the second term in (53) equals

$$M_Q(r) = \frac{1}{10} \cdot \frac{Q^2(r)}{r} \quad (55)$$

which is precisely the flat-space result for the energy stored in the electric field inside a homogeneously charged ball of radius  $r$ .

- (iii) At this stage this manifold of solutions has more free parameters (four) than the charged Nariai solution (three). But additional dependencies will be imposed on the former by the junction conditions, as we will discuss in the following section 5.
- (iv) The uncharged solutions, which are clearly obtained by integrating all differential equations after setting  $\beta = 0$ , are also obtained from our solutions in the limit  $\beta \rightarrow 0$ . We note that this would not be true if we had chosen the Ansatz of [15]. However, in taking the limit  $\beta \rightarrow 0$  one must be careful with the boundary conditions which may also depend on  $\beta$ . We will demonstrate this for our special case of the Nariai spacetime in section 6 in detail.

## 5. Two-mass solution

### 5.1. Junction conditions

Now we wish to combine these solutions to a single spacetime by gluing them along certain boundaries, thereby allowing for specific discontinuities which are restricted by the condition that there shall be no surface layers along the identified surfaces in the newly constructed spacetime. Like in electrodynamics, this results in junction conditions the precise form of which were worked out by several people [16, 8, 12], a text-book presentation being given in §21.13 of [18]. In general, these conditions state that the induced metrics  $\mathbf{h}$  and extrinsic curvatures  $\mathbf{K}$  (essentially corresponding to the normal derivatives of the induced metrics) of the hypersurfaces that are to be identified have to coincide. In our case, each boundary is the history of an  $SO(3)$ -orbit, i.e. it is a timelike surface of topology  $\mathbb{R} \times S^2$ . In that special case the junction conditions can also be given an alternative form [7], part of which states that the areal radii and the Misner-Sharp masses must coincide.

In charged-Nariai spacetime, at each instant in time, the boundary surfaces are located at  $z^\pm$  with  $z^+ > 0$  and  $z^- < 0$  and inward pointing normal 1-form  $\mathbf{N}_N^\pm = \mp \mathbf{d}z$ . We will embed each star along its surface  $\chi_b$ , defined by the vanishing of the pressure  $P(\chi_b) = 0$ , into charged-Nariai spacetime. Here, the outward pointing normal 1-form is

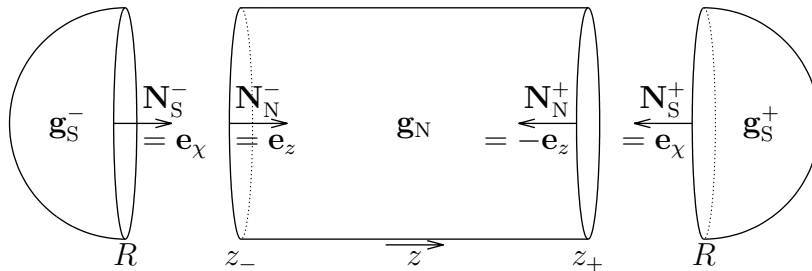


Figure 1: Embedding two stars into Nariai spacetime (schematic)

$\mathbf{N}_S^\pm = \frac{1}{\sqrt{\beta(\mathcal{R}_S^2 \sin^2 \chi + \mathcal{S}^2)}} \mathbf{d}\chi$ . For a schematic representation see figure 1. For the induced metrics we obtain

$$\mathbf{h}_N = -\cos^2 \left( \frac{\lambda z^\pm}{\mathcal{R}_N} \right) \mathbf{d}t^2 + \mathcal{R}_N^2 \mathbf{d}\Omega^2, \quad (56a)$$

$$\mathbf{h}_S^\pm = -\mathcal{A}^\pm \frac{F_b^2}{2\sqrt{11\beta F_b + 1}} \mathbf{d}t^2 + \mathcal{R}_S^2 \sin^2 \chi_b \mathbf{d}\Omega^2, \quad (56b)$$

and for the extrinsic curvatures

$$\mathbf{K}_N = \pm \frac{\lambda}{\mathcal{R}_N} \cos \left( \frac{\lambda z^\pm}{\mathcal{R}_N} \right) \sin \left( \frac{\lambda z^\pm}{\mathcal{R}_N} \right) \mathbf{d}t^2, \quad (57a)$$

$$\mathbf{K}_S^\pm = \sqrt{\beta(\mathcal{R}_S^2 \sin^2 \chi + \mathcal{S}^2)} \left( \mathcal{A}^\pm \frac{F_b(\sqrt{11\beta F_b + 1})F'_b}{(2\sqrt{11\beta F_b + 1})^2} \mathbf{d}t^2 - \mathcal{R}_S^2 \sin \chi_b \cos \chi_b \mathbf{d}\Omega^2 \right), \quad (57b)$$

where  $F_b = F(\chi_b)$ .

Comparing the spatial parts of the extrinsic curvatures we immediately get  $\chi_b = \frac{\pi}{2}$ , so that the star's surface is precisely the equator of the 3-sphere. Equality of the spatial parts of the induced metric then tells us that the radii of the charged-Nariai spacetime  $\mathcal{R}_N$  and the star  $\mathcal{R}_S$  must be the same, hence  $\mathcal{R}_N = \mathcal{R}_S =: \mathcal{R}$ . Clearly, these results had to be expected on geometric grounds. These two conditions also ensure that each star has the mass  $M = \frac{\mathcal{R}}{2G}$ , as demanded by the charged-Nariai metric. Using the formula (50) for the derivative of  $F$  we can simplify the expression for the extrinsic curvature of the star to

$$\mathbf{K}_S^\pm = \mathcal{A}^\pm \mathcal{R} F_b \frac{\sqrt{11\beta F_b + 1}}{2\sqrt{11\beta F_b + 1}} \mathbf{d}t^2. \quad (58)$$

We already know from section 3 that the star at  $z^+$  must have the charge  $Q^+(\mathcal{R}) = q$  and the other one  $Q^-(\mathcal{R}) = -q$  so that  $5\beta\mathcal{R}^6 = Gq^2$ . Since the pressure vanishes at the surface,  $P(\chi_b) = 0$ , we get from (52) an expression for the cosmological constant

$$\Lambda = \alpha - 4\beta\mathcal{R}^2. \quad (59)$$

This condition is not independent as it is also a consequence of (15a) and the definition of  $\mathcal{R}$  by  $1 - \alpha\mathcal{R}^2 - \beta\mathcal{R}^4 = 0$ . However, it allows to derive some useful identities from the definition of the radius  $\mathcal{R}_S$  and  $\alpha$ . Indeed, from

$$2\sqrt{\alpha^2 + 4\beta} = 2\alpha + 4\beta\mathcal{R}^2 = 3\alpha - \Lambda = 2\bar{\rho} \quad (60)$$

we obtain  $\alpha^2 = \bar{\rho}^2 - 4\beta$  and  $2\beta\mathcal{R}^2 = \bar{\rho} - \alpha$ . Using these relations we can rewrite the pressure function as

$$\bar{P}(\chi) = \sqrt{\bar{\rho} - \beta\mathcal{R}^2 \cos^2 \chi} \left( \sqrt{11\beta} + \frac{1}{F(\chi)} \right) \mathcal{R} \cos \chi - 2\beta\mathcal{R}^2 \cos^2 \chi, \quad (61)$$

using  $\beta(\mathcal{R}^2 \sin^2 \chi + \mathcal{S}^2) = \bar{\rho} - \beta\mathcal{R}^2 \cos^2 \chi$ . The constant  $\mathcal{B}$  is related to the central pressure  $p_c = P(0)$  by

$$\bar{p}_c = \sqrt{11\beta} + \frac{1}{F_c} - 2\beta\mathcal{R}^2, \quad (62)$$

where  $F_c = F(0)$ . Solving this for  $\mathcal{B}$  we get

$$\mathcal{B} = \frac{\bar{p}_c + 2\beta\mathcal{R}^2 + \sqrt{11}\beta}{\bar{p}_c + 2\beta\mathcal{R}^2 - \sqrt{11}\beta} \exp\left(-\sqrt{11} \arcsin\left(\sqrt{1 - \frac{4\beta}{\bar{\rho}^2}}\right)\right). \quad (63)$$

The constant  $\mathcal{A}^\pm$  is easily determined by the comparison of the time components of the induced metrics as

$$\mathcal{A}^\pm = \frac{2\sqrt{11}\beta F_b + 1}{F_b^2} \cos^2\left(\frac{\lambda z^\pm}{\mathcal{R}}\right). \quad (64)$$

Finally we compare the time components of the extrinsic curvatures. This leads to the following relation between the central pressure and the position of the star:

$$\pm \tan\left(\frac{\lambda z^\pm}{\mathcal{R}}\right) = \frac{\mathcal{R}^2}{\lambda} \left(\sqrt{11}\beta + \frac{1}{F_b}\right). \quad (65)$$

Altogether the star is described by three independent parameters similar to the Nariai spacetime. We can choose from three independent parameter sets describing the charge  $(q, \sigma, \beta)$ , the mass  $(M, \mathcal{R}, \bar{\rho})$  and their positions  $z^\pm$  or central pressures  $\bar{p}_c$  related by (65).

## 5.2. Allowed parameter sets

In the following we will concentrate on the parameters  $\bar{\rho}$ ,  $\beta$  and  $\bar{p}_c$  and wish to characterise their allowed domains.

The occurrent square root  $\sqrt{\bar{\rho} - \beta\mathcal{R}^2 \cos^2 \chi}$  in the pressure (61) is real for all  $\chi \leq \frac{\pi}{2}$  if the mass density is positive,  $\bar{\rho} > 0$ , and  $\bar{\rho} - \beta\mathcal{R}^2 = \frac{1}{\mathcal{R}^2} > 0$ . The radius  $\mathcal{R}^2 = \frac{\bar{\rho}}{2\beta} \left(1 - \sqrt{1 - \frac{4\beta}{\bar{\rho}^2}}\right)$  as a function of  $\bar{\rho}$  and  $\beta$  is real if  $\beta < \frac{\bar{\rho}^2}{4}$ . These equations imply an upper bound for the modulus of the charge.

From the stress-energy conservation (35) we get for the derivative of the pressure the expression

$$\bar{P}'(\chi) = 15\beta\mathcal{R}^2 \sin \chi \cos \chi - (\bar{\rho} + \bar{P}(\chi)) \frac{\mathcal{R} \sin \chi}{\sqrt{\bar{\rho} - \beta\mathcal{R}^2 \cos^2 \chi}} \left(\sqrt{11}\beta + \frac{1}{F(\chi)}\right), \quad (66)$$

so that the derivative at the boundary is negative

$$\bar{P}'_b = -\mathcal{R}\sqrt{\bar{\rho}} \left(\sqrt{11}\beta + \frac{1}{F_b}\right) = \mp \sqrt{\bar{\rho}} \frac{\lambda}{\mathcal{R}} \tan\left(\frac{\lambda z^\pm}{\mathcal{R}}\right) < 0. \quad (67)$$

This implies that just below the star's surface the pressure is positive. But since the star's surface was defined to be the first zero of the pressure, the pressure must be positive everywhere within the star. In fact, we can show that the pressure is positive if and only if the central pressure is positive. In the neutral case,  $\beta = 0$ , this is immediate since only the negative term in (66) remains so that the pressure must be positive and monotonically decreasing. In the charged case this is a little harder to see. From (61) we obtain the inequality

$$\frac{2\beta\mathcal{R} \cos \chi}{\sqrt{\bar{\rho} - \beta\mathcal{R}^2 \cos^2 \chi}} \leq \sqrt{11}\beta + \frac{1}{F(\chi)}. \quad (68)$$

We first notice that the right-hand side is monotonically increasing in  $\bar{p}_c$  because of  $\frac{d}{d\bar{p}_c} \frac{1}{F(\chi)} > 0$ , as one easily verifies by direct calculation. Therefore, we may set  $\bar{p}_c = 0$ , in which case  $F(\chi)$  is monotonically decreasing in  $\chi$ . Considering (50) for the derivative of  $F(\chi)$  this means that  $2\sqrt{11\beta}F_c + 1 = \frac{2\beta\mathcal{R}^2 + \sqrt{11\beta}}{2\beta\mathcal{R}^2 - \sqrt{11\beta}} < 0$  or  $2\beta\mathcal{R}^2 < \sqrt{11\beta}$  using (62) with  $\bar{p}_c = 0$ . The last inequality can be rewritten as  $1 - \sqrt{1 - 4x} < \sqrt{11x}$  with  $x = \frac{\beta}{\rho^2}$  which is true for  $0 < x \leq \frac{1}{4}$ . Hence we have

$$\sqrt{11\beta} + \frac{1}{F(\chi)} \geq \sqrt{11\beta} + \frac{1}{F_c} = 2\beta\mathcal{R}^2. \quad (69)$$

Since the left side is monotonically decreasing in  $\chi$ , which can again be checked easily by direct calculation, we have

$$\frac{2\beta\mathcal{R} \cos \chi}{\sqrt{\bar{\rho} - \beta\mathcal{R}^2 \cos^2 \chi}} \leq 2\beta\mathcal{R}^2. \quad (70)$$

In total we thus get

$$\frac{2\beta\mathcal{R} \cos \chi}{\sqrt{\bar{\rho} - \beta\mathcal{R}^2 \cos^2 \chi}} \leq 2\beta\mathcal{R}^2 \leq \sqrt{11\beta} + \frac{1}{F(\chi)}, \quad (71)$$

showing the desired result that the pressure is positive everywhere within in the star. Note that it is not excluded that the pressure increases near the centre in an outward direction because there we have

$$\bar{P}(\chi) = \bar{p}_c + \frac{1}{2} \left( 15\beta\mathcal{R}^2 - \mathcal{R} \frac{(\bar{\rho} + \bar{p}_c)}{\sqrt{\bar{\rho}}} (\bar{p}_c + 2\beta\mathcal{R}^2) \right) \chi^2 + \mathcal{O}(\chi^3). \quad (72)$$

Having shown that the pressure is bounded below by zero we next wish to show that it is also bounded above by the density. As already stated, the pressure may assume its maximal value off the centre. If the maximum is at the centre, the pressure must monotonically decrease as we move away from the centre towards the surface. Hence  $\bar{P}(\chi) \leq \bar{p}_c$ . Assuming there exists a maximum at  $\hat{\chi} \in (0, \pi/2)$  bigger than the central pressure, we have 0

$$\bar{P}'(\hat{\chi}) = 15\beta\mathcal{R}^2 \sin \hat{\chi} \cos \hat{\chi} - (\bar{\rho} + \bar{P}(\hat{\chi})) \frac{\mathcal{R} \sin \hat{\chi}}{\sqrt{\bar{\rho} - \beta\mathcal{R}^2 \cos^2 \hat{\chi}}} \left( \sqrt{11\beta} + \frac{1}{F(\hat{\chi})} \right) = 0. \quad (73)$$

If we insert

$$\hat{P} := \bar{P}(\hat{\chi}) = \sqrt{\bar{\rho} - \beta\mathcal{R}^2 \cos^2 \hat{\chi}} \left( \sqrt{11\beta} + \frac{1}{F(\hat{\chi})} \right) \mathcal{R} \cos \hat{\chi} - 2\beta\mathcal{R}^2 \cos^2 \hat{\chi} \quad (74)$$

we obtain

$$15\beta\mathcal{R}^2 \cos^2 \hat{\chi} (\bar{\rho} - \beta\mathcal{R}^2 \cos^2 \hat{\chi}) = (\bar{\rho} + \hat{P}) (\hat{P} + 2\beta\mathcal{R}^2 \cos^2 \hat{\chi}). \quad (75)$$

Using the abbreviation  $\xi = \beta\mathcal{R}^2 \cos^2 \hat{\chi}$  we can rewrite this equation as

$$\hat{P}^2 + (\bar{\rho} + 2\xi) \hat{P} + (15\xi - 13\bar{\rho}) \xi = 0 \quad (76)$$

with the positive solution

$$\hat{P} = -\left(\frac{\bar{\rho}}{2} + \xi\right) + \sqrt{\left(\frac{\bar{\rho}}{2} + \xi\right)^2 + (13\bar{\rho} - 15\xi) \xi}. \quad (77)$$



As we will see below, the relevant sector is given by  $0 \leq \beta \leq \frac{10}{121}\bar{\rho}^2$ . Hence we have  $0 \leq \xi \leq \frac{1}{11}\bar{\rho}$ , leading to the desired bound  $\bar{P} \leq \bar{\rho}$ . This is also shown in figure 2.

Since the derivative is bounded from above the pressure never reaches infinity before it decreases. Moreover, if the central pressure is negative there must always be at least one sphere within the star where the pressure either vanishes (contradicting the assumption that the radius is the first zero) or diverges, leading to a discontinuity.

From the previous section we know that the central pressure is related to the position of the star. So not all positions are allowed, as shown in figure 3. For each position there is only an interval of possible charges of the star so that the pressure is positive everywhere. In the other regions the central pressure is negative and discontinuities (lower right region) or further roots of the pressure function (upper left region) occur. Some characteristic pressure distributions are shown in figure 4.

As figure 3 indicates there is a critical upper bound for the modulus of the charge, given by  $\beta_{\text{crit}} < \frac{1}{4}\bar{\rho}^2$ . This is the point where  $\lambda = 0$  or  $\beta = \frac{10}{121}\bar{\rho}^2$  and the charged-Nariai spacetime changes its topology, turning into a cosmological Bertotti-Robinson spacetime. If we keep the central pressure  $\bar{p}_c$  constant and increase the charge we move within the allowed region but  $g_{00}$  tends to zero. The latter implies diverging acceleration of the stationary Killing orbits and hence an instability of the star. Higher charges are also not possible, for the pressure then turns negative. Because of the junction condition

$$\mp \tanh\left(\frac{\lambda z^\pm}{\mathcal{R}}\right) = \frac{\mathcal{R}^2}{\lambda} \left(\sqrt{11\beta} + \frac{1}{F_b}\right) \quad (78)$$

we have  $\sqrt{11\beta} + \frac{1}{F_b} < 0$  and thus  $\bar{P}(\chi) < 0$ . However, it should be possible to embed only one star.

## 6. Neutral Limit

Now we want to consider the neutral limit  $\beta \rightarrow 0$ . For this we have to keep two parameters constant. These will be the mass density  $\bar{\rho}$  and the central pressure  $\bar{p}_c$ .

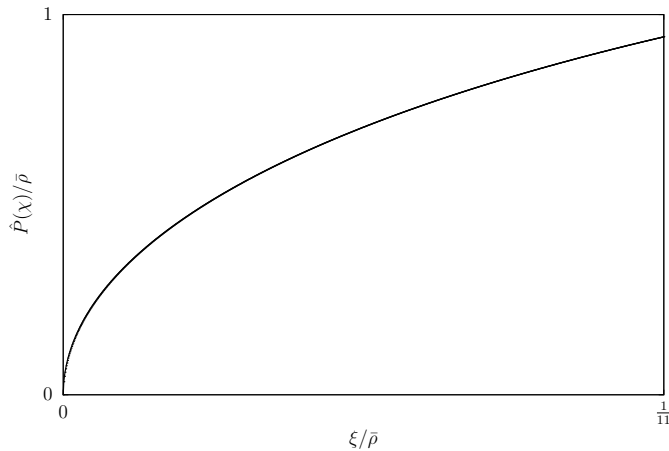


Figure 2: Maximal pressure

At first we consider the charged-Nariai spacetime and start with the radius  $\mathcal{R}$ . Because of

$$2\beta\mathcal{R}^2 = \bar{\rho} \left( 1 - \sqrt{1 - \frac{4\beta}{\bar{\rho}^2}} \right) = \frac{2\beta}{\bar{\rho}} + \frac{2\beta^2}{\bar{\rho}^3} + \mathcal{O}(\beta^3) \quad (79)$$

we obtain  $\mathcal{R}^2 = \frac{1}{\bar{\rho}} + \mathcal{O}(\beta)$ . We can derive the Taylor expansion for the cosmological constant from (59). This gives us

$$\Lambda = \bar{\rho} \left( 3\sqrt{1 - \frac{4\beta}{\bar{\rho}^2}} - 2 \right) = \bar{\rho} - \frac{6\beta}{\bar{\rho}} + \mathcal{O}(\beta^2) \quad (80)$$

and agrees with the expression we would get from (15a). Furthermore we have

$$\lambda^2 = 1 - 10\beta\mathcal{R}^4 = 1 - \frac{10\beta}{\bar{\rho}^2} + \mathcal{O}(\beta^2) \quad (81)$$

and thus  $\lambda = 1 + \mathcal{O}(\beta)$ . Hence the charged Nariai metric reduces to the common Nariai metric [20, 19]

$$\mathbf{g}_N = -\cos^2\left(\frac{z}{\mathcal{R}_0}\right) \mathbf{d}t^2 + \mathbf{d}z^2 + \mathcal{R}_0^2 \mathbf{d}\Omega^2 \quad (82a)$$

$$= \frac{1}{\Lambda_0} (-\cos^2(Z) \mathbf{d}T^2 + \mathbf{d}Z^2 + \mathbf{d}\Omega^2) \quad (82b)$$

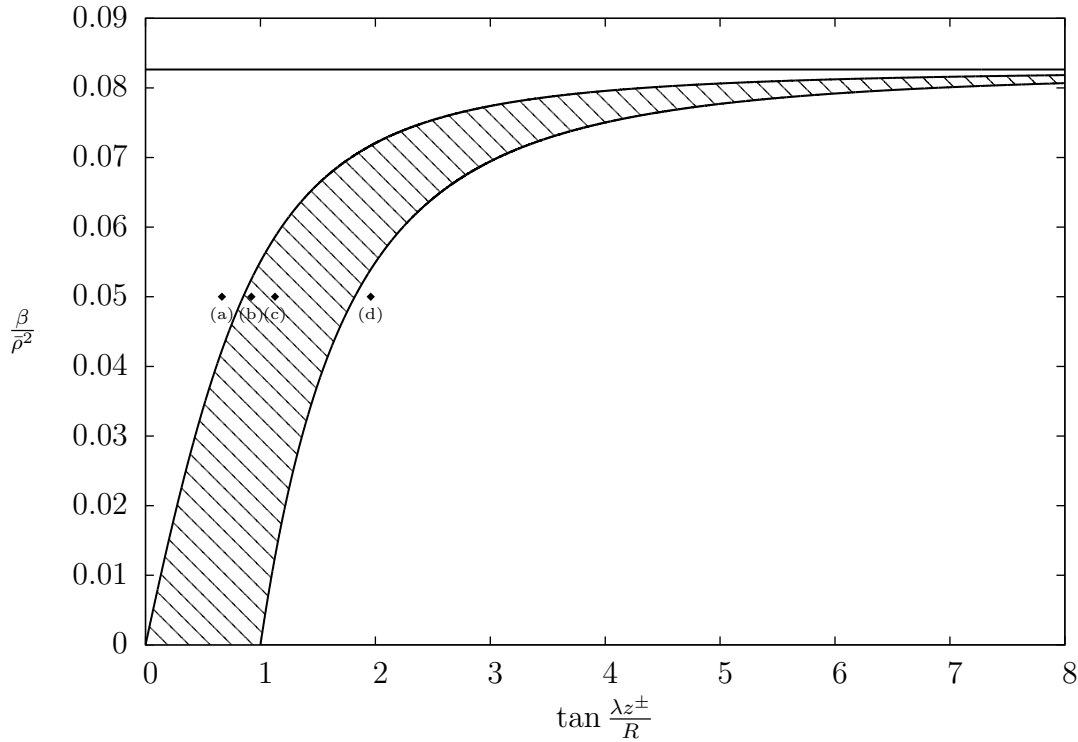


Figure 3: In the ruled region the pressure is positive within the star. The rule corresponds to the critical charge  $\beta/\bar{\rho}^2 = \frac{10}{121}$ . The four marks correspond to the four pressure distributions shown in figure 4.

with  $T = \sqrt{\Lambda_0} t$ ,  $Z = \sqrt{\Lambda_0} z$  and  $\Lambda_0 = 1/\mathcal{R}_0^2 = \bar{\rho}$ .

Now let's turn to the star's metric. We begin with the expansion of the constant  $\mathcal{B}$ . We have for the first part in (63)

$$\frac{\bar{p}_c + 2\beta\mathcal{R}^2 + \sqrt{11\beta}}{\bar{p}_c + 2\beta\mathcal{R}^2 - \sqrt{11\beta}} = 1 + \frac{2}{\bar{p}_c} \sqrt{11\beta} + \mathcal{O}(\beta). \quad (83)$$

For the second part we consider (note:  $\arcsin(1-x) = \frac{\pi}{2} - \sqrt{2x} + \mathcal{O}(x^{3/2})$ )

$$\sqrt{11} \arcsin\left(1 - \frac{2\beta\mathcal{R}^2}{\bar{\rho}} \cos^2 \chi\right) = \sqrt{11} \frac{\pi}{2} - \frac{2}{\bar{\rho}} \sqrt{11\beta} \cos \chi + \mathcal{O}(\beta), \quad (84)$$

so that

$$\exp\left[\sqrt{11} \arcsin\left(1 - \frac{2\beta\mathcal{R}^2}{\bar{\rho}} \cos^2 \chi\right)\right] = \exp\left(\sqrt{11} \frac{\pi}{2}\right) \left[1 + \frac{2}{\bar{\rho}} \sqrt{11\beta} \cos \chi + \mathcal{O}(\beta)\right]. \quad (85)$$

If we set  $\chi = \frac{\pi}{2}$  and invert the expression we can derive the expansion for  $\mathcal{B}$

$$\mathcal{B} = \exp\left(-\sqrt{11} \frac{\pi}{2}\right) \left(1 + \frac{2}{\bar{\rho}} \sqrt{11\beta} \frac{\bar{\rho} + \bar{p}_c}{\bar{p}_c} + \mathcal{O}(\beta)\right). \quad (86)$$

Combining the last two results we can expand the function  $F(\chi)$

$$2\sqrt{11\beta}F(\chi) = \frac{2}{\bar{\rho}} \sqrt{11\beta} \left(\frac{\bar{\rho} + \bar{p}_c}{\bar{p}_c} - \cos \chi\right) + \mathcal{O}(\beta), \quad (87)$$

so that

$$\frac{F^2(\chi)}{2\sqrt{11\beta}F(\chi) + 1} = \frac{1}{\bar{\rho}^2} \left[\frac{\bar{\rho} + \bar{p}_c}{\bar{p}_c} - \cos \chi\right]^2 + \mathcal{O}(\beta^{1/2}). \quad (88)$$

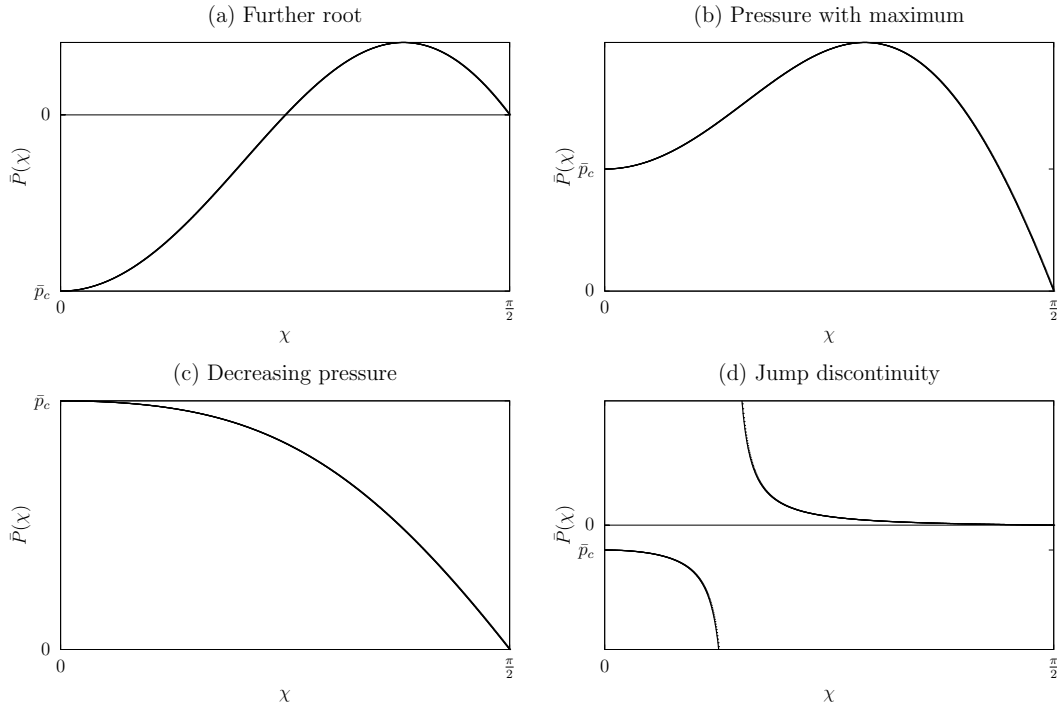


Figure 4: Characteristic pressure distributions (their positions are marked in figure 3)

Before we can determine the metric we need the expansion of  $\mathcal{A}$  and  $z^\pm$ . We get

$$\begin{aligned} z^\pm &= \pm \frac{\mathcal{R}}{\lambda} \arctan \left[ \frac{\mathcal{R}^2}{\lambda} \left( \sqrt{11\beta} + \frac{1}{F_b} \right) \right] \\ &= \pm \left( \frac{1}{\sqrt{\bar{\rho}}} + \mathcal{O}(\beta^{1/2}) \right) \arctan \left[ \frac{\bar{p}_c}{\bar{\rho} + \bar{p}_c} + \mathcal{O}(\beta^{1/2}) \right] \\ &= \pm \frac{1}{\sqrt{\bar{\rho}}} \arctan \left( \frac{\bar{p}_c}{\bar{\rho} + \bar{p}_c} \right) + \mathcal{O}(\beta^{1/2}). \end{aligned} \quad (89)$$

Hence  $z_0^\pm = \pm \mathcal{R}_0 \arctan \frac{\bar{p}_c}{\bar{\rho} + \bar{p}_c}$ . Furthermore we have  $\cos^2 \left( \frac{\lambda z^\pm}{\mathcal{R}} \right) = \cos^2 \left( \frac{z_0^\pm}{\mathcal{R}_0} \right) + \mathcal{O}(\beta^{1/2})$ . Now we can easily calculate the constant  $\mathcal{A}$  from (64)

$$\mathcal{A}^\pm = \bar{\rho}^2 \left[ \frac{\bar{p}_c}{\bar{\rho} + \bar{p}_c} \right]^2 \cos^2 \left( \frac{z_0^\pm}{\mathcal{R}_0} \right) + \mathcal{O}(\beta^{1/2}). \quad (90)$$

Combining all our results, we can expand the metric components

$$g_{00} = \cos^2 \left( \frac{z_0^\pm}{\mathcal{R}_0} \right) \left[ 1 - \frac{\bar{p}_c}{\bar{\rho} + \bar{p}_c} \cos \chi \right]^2 + \mathcal{O}(\beta^{1/2}), \quad (91a)$$

$$g_{11} = \frac{1}{\bar{\rho} - \beta \mathcal{R}^2 \cos^2 \chi} = \frac{1}{\bar{\rho}} + \mathcal{O}(\beta). \quad (91b)$$

Hence the star's metric in neutral limit is

$$\mathbf{g}_S = - \cos^2 \left( \frac{z_0^\pm}{\mathcal{R}_0} \right) \left[ 1 - \frac{\bar{p}_c}{\bar{\rho} + \bar{p}_c} \cos \chi \right]^2 dt^2 + \frac{1}{\bar{\rho}} (d\chi^2 + d\Omega^2). \quad (92)$$

This is an agreement with eq. (3.33) of [4], except for the prefactor which depends on the surrounding spacetime. Upon rescaling the time coordinate it is possible to arrive at the same expression.

Finally we consider the pressure

$$\bar{P}(\chi) = \bar{\rho} \frac{\bar{p}_c \cos \chi}{\bar{\rho} + \bar{p}_c - \bar{p}_c \cos \chi} + \mathcal{O}(\beta^{1/2}), \quad (93)$$

which for  $\beta = 0$  coincides with eq. (3.32) of [4].

## 7. Discussion

In this paper we investigated exact solutions to Einstein's equations which represent two spherically symmetric stars made from an incompressible perfect fluid, possibly with non-vanishing charge density which is constant with respect to the areal volume element. The stars are kept at constant distance and the solution is globally static and spherically symmetric. The topology of the spatial splices of simultaneity is that of a three-sphere, so that the total electric charge must be zero. In fact, the stars have charges of equal modulus and opposite sign, leading to further attraction. The combined gravitational and electric attraction of the stars is balanced by the negative pressure of a positive cosmological constant without causing a cosmological horizon separating the stars. These solutions were not contained in previous analyses and somehow bridge between the results obtained in [25] and those in [4]. Being exact

solutions to Einstein's equations it is clear that they can claim some interest in their own right. In our interpretation of the possible physical significance of these solutions we follow [25], who see the study of exact two-mass solutions as an initial step towards a more rigorous understanding of local geometric structure in inhomogeneous cosmologies, though this admittedly means stressing one's imagination, in particular given the ratios of the cosmological constant to the mass density involved in our solutions. However, whereas it seems now clear that the spin-spin-interaction of aligned (sub-extremal) Kerr black-holes cannot balance their gravitational attraction (so as to result in a stationary exterior spacetime) [21], it is interesting to see what a cosmological constant can do. This, to us, motivates further investigations in this direction and perhaps combine results.

## Acknowledgments

Hospitality and support of the Center of Applied Space Technology and Microgravity (ZARM) at Bremen is gratefully acknowledged. Micheal Fennen was supported by a Ph.D. grant of the German Research Foundation (DFG) within its Research Training Group no.1620 *Models of Gravity*. Domenico Giulini was supported by the Cluster of Excellence *Centre for Quantum Engineering and Space-Time Research* of the DFG.

## Appendix B. Curvatures of Nariai-Type metrics

We consider metrics of the form (2) and wish to calculate the curvature coefficients with respect to the orthonormal tetrad (3a)-(3d). This we do by solving Cartan's first structure equation (expressing vanishing torsion)

$$\mathbf{d}\theta^a + \omega^a_b \wedge \theta^b = 0, \quad (\text{B.1})$$

for the connection 1-forms  $\omega^a_b$ . This gives a unique solution for metric-compatible connections, which satisfy  $g_{ac}\omega^c_b = -g_{bc}\omega^c_a$ . Note that here all indices refer to components with respect to the orthonormal tetrad, so that  $g_{11} = g_{22} = g_{33} = -g_{00} = 1$  and  $g_{ab} = 0$  for  $a \neq b$ . Using (3a)-(3d) a straightforward calculation gives

$$\omega^0_1 = \frac{a'}{a} \theta^0 = a' \mathbf{d}t, \quad (\text{B.2})$$

$$\omega^0_2 = \omega^0_3 = 0, \quad (\text{B.3})$$

$$\omega^1_2 = -\frac{R'}{R} \theta^2 = -R' \mathbf{d}\vartheta, \quad (\text{B.4})$$

$$\omega^1_3 = -\frac{R'}{R} \theta^3 = -R' \sin \vartheta \mathbf{d}\varphi \quad (\text{B.5})$$

$$\omega^2_3 = -\frac{\cot \vartheta}{R} \theta^3 = -\cos \vartheta \mathbf{d}\varphi. \quad (\text{B.6})$$

From these the curvature 2-forms follow by Cartan's second structure equation:

$$\Omega^a_b = \frac{1}{2} R^a_{bcd} \theta^c \wedge \theta^d = \mathbf{d}\omega^a_b + \omega^a_c \wedge \omega^c_b \quad (\text{B.7})$$

and thus, in turn, the non-vanishing components of the (totally covariant) Riemann tensor  $R_{abcd} = g_{an}R^n_{bcd}$ :

$$R_{01\,01} = \frac{a''}{a}, \quad (B.8)$$

$$R_{02\,02} = R_{03\,03} = \frac{a'R'}{aR}, \quad (B.9)$$

$$R_{12\,12} = R_{13\,13} = -\frac{R''}{R}, \quad (B.10)$$

$$R_{23\,23} = \frac{1 - R'^2}{R^2}. \quad (B.11)$$

The components of the Ricci tensor follow from

$$R_{ab} = -R_{0a\,0b} + R_{1a\,1b} + R_{2a\,2b} + R_{3a\,3b}, \quad (B.12)$$

of which the non-vanishing ones are

$$R_{00} = \frac{a''}{a} + 2\frac{a'R'}{aR}, \quad (B.13)$$

$$R_{11} = -\frac{a''}{a} - 2\frac{R''}{R}, \quad (B.14)$$

$$R_{22} = R_{33} = \frac{1 - R'^2}{R^2} - \frac{a'R'}{aR} - \frac{R''}{R}. \quad (B.15)$$

This gives the scalar curvature

$$\begin{aligned} R &= -R_{00} + R_{11} + R_{22} + R_{33} \\ &= 2 \left[ \frac{1 - R'^2}{R^2} - \frac{a''}{a} - 2\frac{R''}{R} - 2 - \frac{a'R'}{aR} \right]. \end{aligned} \quad (B.16)$$

Finally, (B.13)-(B.15) and (B.16) give the non-vanishing components of the Einstein tensor:

$$G_{00} = R_{00} + \frac{R}{2} = -2\frac{R''}{R} + \frac{1 - R'^2}{R^2}, \quad (B.17)$$

$$G_{11} = R_{11} - \frac{R}{2} = 2\frac{a'R'}{aR} - \frac{1 - R'^2}{R^2}, \quad (B.18)$$

$$G_{22} = G_{33} = R_{22} - \frac{R}{2} = \frac{a''}{a} + \frac{R''}{R} + \frac{a'R'}{aR}. \quad (B.19)$$

- [1] Rudolf Bach and Hermann Weyl. Neue Lösungen der Einsteinschen Gravitationsgleichungen. B. Explizite Aufstellung statischer axialsymmetrischer Felder. Mit einem Zusatz über das statische Zweikörperproblem. *Mathematische Zeitschrift*, 13:134–145, 1922.
- [2] Bruno Bertotti. Uniform electromagnetic field in the theory of general relativity. *Physical Review*, 116(5):1331–1333, 1959.
- [3] Christian G. Böhmer. General relativistic static fluid solutions with cosmological constant. arXiv:gr-qc/0308057v3, 2003. Diploma Thesis, University of Potsdam.
- [4] Christian G. Böhmer. Eleven spherically symmetric constant density solutions with cosmological constant. *General Relativity and Gravitation*, 36(5):1039–1054, 2004.
- [5] Christian G. Böhmer and Gyula Fodor. Perfect fluid spheres with cosmological constant. *Physical Review D*, 77(6):064008 (12 pages), 2008.
- [6] Christian G. Böhmer and Atifah Mussa. Charged perfect fluids in the presence of a cosmological constant. *General Relativity and Gravitation*, 43(11):3033–3046, 2011.
- [7] Matteo Carrera and Domenico Giulini. Influence of global cosmological expansion on local dynamics and kinematics. *Reviews of Modern Physics*, 82(1):169–208, 2010.
- [8] Georges Darmois. Les équations de la gravité einsteinienne. *Mémoires des sciences mathématiques*, 25:1–48, 1927.
- [9] Jerry B. Griffiths and Jiří Podolský. *Exact Space-Times in Einstein's General Relativity*. Cambridge University Press, Cambridge, 2009.
- [10] Stephen W. Hawking. Gravitational radiation in an expanding universe. *Journal of Mathematical Physics*, 9(4):598–604, 1968.
- [11] Stephen W. Hawking and George F.R. Ellis. *The Large Scale Structure of Spacetime*. Cambridge University Press, Cambridge, 1973.
- [12] Werner Israel. Singular hypersurfaces and thin shells in general relativity. *Il Nuovo Cimento*, 44 B(1):1–14, 1966. Errata ibid **48B**(2), 463.
- [13] Diana Kormos Buchwald, József Illy, Ze'ev Rosenkranz, and Tilman Sauer, editors. *The Collected Papers of Albert Einstein, Vol. 13*. Princeton University Press, Princeton, New Jersey, 2012.
- [14] Friedrich Kottler. Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie. *Annalen der Physik*, 56(14):401–462, 1918.
- [15] F. Kyle, C. and W. Martin, A. Self-energy considerations in general relativity and the exact field of charge and mass distributions. *Il Nuovo Cimento A*, 50(3):583–604, 1967.
- [16] Kornel Lanczos. Flächenhafte Verteilung der Materie in der Einsteinschen Gravitationstheorie. *Annalen der Physik*, 379(14):518–540, 1924.
- [17] Charles W. Misner and David H. Sharp. Relativistic equations for adiabatic, spherically symmetric gravitational collapse. *Physical Review*, 136(2B):B571–B576, 1964.
- [18] Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler. *Gravitation*. W.H. Freeman and Company, New York, 1973.
- [19] Hidekazu Nariai. On a new cosmological solution of Einstein's field equation of gravitation. *General Relativity and Gravitation*, 31(6):963–971, 1999. Originally published in *The Science Reports of the Tohoku University Series I*, vol. XXXV, No. 1 (1951), p. 46–57.
- [20] Hidekazu Nariai. On some static solutions of Einstein's gravitational field equations in a spherically symmetric case. *General Relativity and Gravitation*, 31(6):951–961, 1999. Originally published in *The Science Reports of the Tohoku University Series I*, vol. XXXIV, No. 3 (1950), p. 160–167.
- [21] Gernot Neugebauer and Jörg Henning. Non-existence of stationary two-black-hole configurations. *General Relativity and Gravitation*, 41(9):2113–2130, 2009.
- [22] Karl Schwarzschild. Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie. *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, pages 424–434, 1916. Publically available at [http://de.wikisource.org/wiki/Karl\\_Schwarzschild](http://de.wikisource.org/wiki/Karl_Schwarzschild).
- [23] Norbert Straumann. *General Relativity*. Springer Verlag, Berlin, second edition, 2013.
- [24] Erich Trefftz. Das statische Gravitationsfeld zweier Massenpunkte in der Einsteinschen Theorie.

*Mathematische Annalen*, 86(3-4):317–326, 1922.

- [25] Jean-Philippe Uzan, George F.R. Ellis, and Julien Larena. A two-mass expanding exact space-time solution. *General Relativity and Gravitation*, 43(1):191–205, 2011.
- [26] Hermann Weyl. Über die statischen kugelsymmetrischen Lösungen von Einsteins «kosmologischen» Gravitationsgleichungen. *Physikalische Zeitschrift*, 20(7):31–34, 1919.
- [27] Hermann Weyl. *Raum Zeit Materie*. Springer Verlag, Berlin, 8th edition, 1991. Based on the 1923 5th edition, with additional appendices, edited and annotated by Jürgen Ehlers.